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Abstract: Discriminatory Processor Sharing policy introduced by Kleinrock is of a great interest in many application areas, including telecommunications, web applications and TCP flow modelling. Under the DPS policy the job priority is controlled by a vector of weights. Varying the vector of weights, it is possible to modify the service rates of the jobs and optimize system characteristics. In the present paper we present results concerning the comparison of two DPS policies with different weight vectors. We show the monotonicity of the expected sojourn time of the system depending on the weight vector under certain condition on the system. Namely, the system has to consist of classes with means which are quite different from each other. For the classes with similar means we suggest to select the same weights.

Key-words: Discriminatory Processor Sharing, exponential service times, optimization.

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Comparaison des politiques DPS

Résumé : L'ordre de service DPS (Discriminatory Processor Sharing) qui était introduit par Kleinrock est un problème très intéressant et peut être appliqué dans beaucoup de domaines comme les télécommunications, les applications web et la modélisation de flux TCP. Avec le DPS, les jobs qui viennent dans le système sont contrôlés par un vecteur de poids. En modifiant le vecteur de poids, il est possible de contrôler les taux de service des jobs, donner la priorité à certaines classes de jobs et optimiser certaines caractéristiques du système. Le problème du choix des poids est donc très important et très difficile en raison de la complexité du système. Dans le présent papier, nous comparons deux politiques DPS avec les vecteurs de poids différents et nous présentons des résultats sur la monotonicité du temps moyen de service du système en fonction du vecteur de poids, sous certaines conditions sur le système. Le système devrait consister en plusieurs classes avec des moyennes très différentes. Pour les classes qui ont une moyenne très proche il faut choisir les meme poids.

Mots-clés : Discriminatory Processor Sharing, le temp de service exponentielle, optimisation.

1 Introduction

The Discriminatory Processor Sharing (DPS) policy was introduced by Kleinrock [11]. Under the DPS policy jobs are organized in classes, which share a single server. The capacity that each class obtains depends on the number of jobs currently presented in all classes. All jobs present in the system are served simultaneously at rates controlled by the vector of weights $g_k > 0$, $k = 1, \ldots, M\}$, where M is the number of classes. If there are N_j jobs in class j, then each job of this class is served with the rate $g_j / \sum_{k=1}^{M} g_k N_k$. When all weights are equal, DPS system is equivalent to the standard PS policy.

The DPS policy model has recently received a lot of attention due to its wide range of application. For example, DPS could be applied to model flow level sharing of TCP flows with different flow characteristics such as different RTTs and packet loss probabilities. DPS also provides a natural approach to model the weighted round-robin discipline, which is used in operating systems for task scheduling. In the Internet one can imagine the situation that servers provide different service according to the payment rates. For more applications of DPS in communication networks see [2], [4], [5], [7], [12].

Varying DPS weights it is possible to give priority to different classes at the expense of others, control their instantaneous service rates and optimize different system characteristics as mean sojourn time and so on. So, the proper weight selection is an important task, which is not easy to solve because of the model's complexity.

The previously obtained results on DPS model are the following. Kleinrock in [11] was first studying DPS. Then the paper of Fayolle et al. [6] provided results for the DPS model. For the exponentially distributed required service times the authors obtained the expression of the expected sojourn time as a solution of a system of linear equations. The authors show that independently of the weights the slowdown for the expected conditional response time under the DPS policy tends to the constant slowdown of the PS policy as the service requirements increases to infinity.

Rege and Sengupta in [13] proved a decomposition theorem for the conditional sojourn time. For exponential service time distributions in [14] they obtained higher moments of the queue length distribution as the solutions of linear equations system and also provided a theorem for the heavy-traffic regime. Van Kessel et al. in [8], [10] study the performance of DPS in an asymptotic regime using time scaling. For general distributions of the required service times the approximation analysis was carried out by Guo and Matta in [7]. Altman et al. [2] study the behavior of the DPS policy in overload. Most of the results obtained for the DPS queue were collected together in the survey paper of Altman et al. [1].

Avrachenkov et al. in [3] proved that the mean unconditional response time of each class is finite under the usual stability condition. They determine the asymptote of the conditional sojourn time for each class assuming finite service time distribution with finite variance.

The problem of weights selection in the DPS policy when the job size distributions are exponential was studied by Avrachenkov et al. in [3] and by Kim and Kim in [10]. In [10] it was shown that the DPS policy reduces the expected sojourn time in comparison with PS policy when the weights increase in the opposite order with the means of job classes. Also in [10] the authors formulate a conjecture about the monotonicity of the expected sojourn time of the DPS policy. The idea of conjecture is that comparing two DPS policies, one which has a weight vector closer to the optimal policy provided by $c\mu$-rule, see [15], has smaller expected sojourn time. Using the method described in [10] in the present paper we prove this conjecture with some restrictions on the system parameters. The restrictions on the system are such that the result is true for systems for which the values of the job size distribution means are very different from each other. The restriction can be overcome by setting the same weights for the classes, which have similar means. The condition on means is a sufficient, but not a necessary condition. It becomes less strict when the system is less loaded.

The paper is organized as follows. In Section 2 we give general definitions of the DPS policy and formulate the problem of expected sojourn time minimization. In Section 3 we formulate the main Theorem and prove it. In Section 4 we give the numerical results. Some technical proofs can be found in the Appendix.

2 Previous results and problem formulation

We consider the Discriminatory Processor Sharing (DPS) model. All jobs are organized in M classes and share a single server. Jobs of class $k = 1, \dots, M$ arrive with a Poisson process with rate λ_k and have required service-time distribution $F_k(x) = 1 - e^{-\mu_k x}$ with mean $1/\mu_k$. The load of the system is $\rho = \sum_{k=1}^{M} \rho_k$ and $\rho_k = \lambda_k/\mu_k$, $k = 1, \dots, M$. We consider that the system is stable, $\rho < 1$. Let us denote $\lambda = \sum_{k=1}^{M} \lambda_k$.

The state of the system is controlled by a vector of weights $g = (g_1, \dots, g_M)$, which denotes the priority for the job classes. If in the class k there are currently N_k jobs, then each job of class k is served with the rate equal to $g_j / \sum_{k=1}^{M} g_k N_k$, which depends on the current system state, or on the number of jobs in each class.

Let $\overline{T}^{DPS}$ be the expected sojourn time of the DPS system. We have

$$\overline{T}^{DPS} = \sum_{k=1}^{M} \frac{\lambda_k}{\lambda} \overline{T}_k,$$

where $\overline{T}_k$ are expected sojourn times for class k. The expressions for the expected sojourn times $\overline{T}_k$, $k = 1, \dots, M$ can be found as a solution of the system of linear equations, see [6],

$$\overline{T}_k \left(1 - \sum_{j=1}^{M} \frac{\lambda_j g_j}{\mu_j g_j + \mu_k g_k} \right) - \sum_{j=1}^{M} \frac{\lambda_j g_j \overline{T}_j}{\mu_j g_j + \mu_k g_k} = \frac{1}{\mu_k}, \quad k = 1, \dots, M. \tag{1}$$

Let us notice that for the standard Processor Sharing system

$$\overline{T}^{PS} = \frac{m}{1-\rho}.$$

One of the problems when studying DPS is to minimize the expected sojourn time $\overline{T}^{DPS}$ with some weight selection. Namely, find g^* such as

$$\overline{T}^{DPS}(g^*) = \min_{g} \overline{T}^{DPS}(g).$$

This is a general problem and to simplify it the following subcase is considered. To find a set G such that

$$\overline{T}^{DPS}(g^*) \leq \overline{T}^{PS}, \quad \forall g^* \in G. \tag{2}$$

For the case when job size distributions are exponential the solution of (2) is given by Kim and Kim in [10] and is as follows. If the means of the classes are such as $\mu_1 \geq \mu_2 \geq \dots \geq \mu_M$, then G consists of all such vectors which satisfy

$$G = \{g |\ g_1 \geq g_2 \geq \dots \geq g_M\}.$$

Using the approach of [10] we solve more general problem about the monotonicity of the expected sojourn time in the DPS system, which we formulate in the following section as Theorem 1.

3 Expected sojourn time monotonicity

Let us formulate and prove the following Theorem.

Theorem 1. *Let the job size distribution for every class be exponential with mean* μ_i*,* $i = 1, \ldots, M$ *and we enumerate them in the following way*

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_M.$$

Let us consider two different weight policies for the DPS system, which we denote as α *and* β*. Let* $\alpha, \beta \in G$*, or*

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_M,$$
$$\beta_1 \geq \beta_2 \geq \ldots \geq \beta_M.$$

The expected sojourn time of the DPS policies with weight vectors α *and* β *satisfies*

$$\overline{T}^{DPS}(\alpha) \leq \overline{T}^{DPS}(\beta), \tag{3}$$

if the weights α *and* β *are such that:*

$$\frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\beta_{i+1}}{\beta_i}, \quad i = 1, \ldots, M-1, \tag{4}$$

and the following restriction is satisfied:

$$\frac{\mu_{j+1}}{\mu_j} \leq 1 - \rho, \tag{5}$$

for every $j = 1, \ldots, M$*.*

Remark 2. *If for some classes* j *and* $j+1$ *condition (5) is not satisfied, then in practice, by choosing the weights of these classes to be equal, we can still use Theorem 1. Namely, for classes such as* $\frac{\mu_{j+1}}{\mu_j} > 1 - \rho$*, we suggest to set* $\alpha_{j+1} = \alpha_j$ *and* $\beta_{j+1} = \beta_j$*.*

Remark 3. *Theorem 1 shows that the expected sojourn time* $\overline{T}^{DPS}(g)$ *is monotonous according to the selection of weight vector* g*. The closer is the weight vector to the optimal policy, provided by* $c\mu$*-rule, the smaller is the expected sojourn time. This is shown by the condition (4), which shows that vector* α *is closer to the optimal* $c\mu$*-rule policy then vector* β*.*

Theorem 1 is proved with restriction (5). This restriction is a sufficient and not a necessary condition on system parameters. It shows that the means of the job classes have to be quite different from each other. This restriction can be overcome, giving the same weights to the job classes, which mean values are similar. Condition (5) becomes less strict as the system becomes less loaded.

To prove Theorem 1 let us first give some notations and prove additional Lemmas.

Let us rewrite linear system (1) in the matrix form. Let $\overline{T}^{(g)} = [\overline{T}_1^{(g)}, \ldots, \overline{T}_M^{(g)}]^T$ be the vector of $\overline{T}_k^{(g)}$, $k = 1, \ldots, M$. Here by $[\,]^T$ we mean transpose sign, so $[\,]^T$ is a vector. By $[\,]^{(g)}$ we note that this element depends on the weight vector selection $g \in G$. Let us consider that later in the paper vectors $g, \alpha, \beta \in G$, if the opposite is not noticed. Let define matrices $A^{(g)}$ and $D^{(g)}$ in the following way.

$$A^{(g)} = \begin{pmatrix} \frac{\lambda_1 g_1}{\mu_1 g_1 + \mu_1 g_1} & \frac{\lambda_2 g_2}{\mu_1 g_1 + \mu_2 g_2} & \ldots & \frac{\lambda_M g_M}{\mu_1 g_1 + \mu_M g_M} \\ \frac{\lambda_1 g_1}{\mu_2 g_2 + \mu_1 g_1} & \frac{\lambda_2 g_2}{\mu_2 g_2 + \mu_2 g_2} & \ldots & \frac{\lambda_M g_M}{\mu_2 g_2 + \mu_M g_M} \\ \ldots & & & \\ \frac{\lambda_1 g_1}{\mu_M g_M + \mu_1 g_1} & \frac{\lambda_2 g_2}{\mu_M g_M + \mu_2 g_2} & \ldots & \frac{\lambda_M g_M}{\mu_M g_M + \mu_M g_M} \end{pmatrix} \tag{6}$$

$$D^{(g)} = \begin{pmatrix} \sum_i \frac{\lambda_i g_i}{\mu_1 g_1 + \mu_i g_i} & 0 & \ldots & 0 \\ 0 & \sum_i \frac{\lambda_i g_i}{\mu_2 g_2 + \mu_i g_i} & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & \sum_i \frac{\lambda_i g_i}{\mu_M g_M + \mu_i g_i} \end{pmatrix} \tag{7}$$

Then (1) becomes

$$(E - D^{(g)} - A^{(g)})\overline{T}^{(g)} = \left[\frac{1}{\mu_1} \cdots \cdot \frac{1}{\mu_M}\right]^T. \tag{8}$$

We need to find the expected sojourn time of the DPS system $\overline{T}^{DPS}(g)$. According to the definition of $\overline{T}^{DPS}(g)$ and equation (8) we have

$$\overline{T}^{DPS}(g) = \frac{1}{\lambda}[\lambda_1, \dots, \lambda_M]\overline{T}^{(g)} = \frac{1}{\lambda}[\lambda_1, \dots, \lambda_M](E - D^{(g)} - A^{(g)})^{-1}\left[\frac{1}{\mu_1}, \dots, \frac{1}{\mu_M}\right]^T. \tag{9}$$

Let us consider the case when $\lambda_i = 1$ for $i = 1, \dots, M$. This results can be extended for the case when λ_i are different, we prove it following the approach of [10] in Proposition 10 at the end of the current Section. Equation (9) becomes

$$\overline{T}^{DPS}(g) \quad = \quad \underline{1}'(E - D^{(g)} - A^{(g)})^{-1}\left[\rho_1, \dots, \rho_M\right]^T \lambda^{-1}. \tag{10}$$

Let us give the following notations.

$$\sigma_{ij}^{(g)} = \frac{g_j}{\mu_i g_i + \mu_j g_j}.$$

Then $\sigma_{ij}^{(g)}$ have the following properties.

Lemma 4. *$\sigma_{ij}^{(g)}$ and $\sigma_{ji}^{(g)}$ satisfy*

$$\begin{aligned} &\sigma_{ij}^{(g)} g_i = \sigma_{ji}^{(g)} g_j, \\ &\frac{\sigma_{ij}^{(g)}}{\mu_i} + \frac{\sigma_{ji}^{(g)}}{\mu_j} = \frac{1}{\mu_i \mu_j}. \end{aligned} \tag{11}$$

Proof. Follows from the definition of $\sigma_{ij}^{(g)}$. □

Then matrices $A^{(g)}$ and $D^{(g)}$ given by (6) and (7) can be rewritten in the terms of $\sigma_{ij}^{(g)}$.

$$\begin{aligned} &A_{i,j}^{(g)} = \sigma_{ij}^{(g)}, \quad i, j = 1, \dots, M, \\ &D_{i,i}^{(g)} = \sum_j \sigma_{ij}^{(g)}, \quad i = 1, \dots, M, \\ &D_{i,j}^{(g)} = 0, \quad i, j = 1, \dots, M, \quad i \neq j. \end{aligned}$$

For weight vectors α, β the following Lemma is true.

Lemma 5. *If α and β satisfy (4), then*

$$\frac{\alpha_j}{\alpha_i} \leq \frac{\beta_j}{\beta_i}, \quad i = 1, \dots, M-1, \quad \forall j \geq i. \tag{12}$$

Proof. Let us notice that if $a < b$ and $c < d$, then $ac < bd$ when a, b, c, d are positive. Also if $j > i$ then there exist such $l > 0$ that $j = i + l$. Then

$$\frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\beta_{i+1}}{\beta_i}, \quad \frac{\alpha_{i+2}}{\alpha_{i+1}} \leq \frac{\beta_{i+2}}{\beta_{i+1}}, \quad \dots \quad \frac{\alpha_{i+l}}{\alpha_{i+l-1}} \leq \frac{\beta_{i+l}}{\beta_{i+l-1}}, \quad i = 1, \dots, M-2.$$

Multiplying left and right parts of the previous inequalities we get the following:

$$\frac{\alpha_{i+l}}{\alpha_i} \leq \frac{\beta_{i+l}}{\beta_i}, \quad i = 1, \dots, M-2,$$

which proves Lemma 5. □

Lemma 6. *If α and β satisfy (12), then*

$$\sigma_{ij}^{(\alpha)} \le \sigma_{ij}^{(\beta)}, \quad i \le j,$$
$$\sigma_{ij}^{(\alpha)} \ge \sigma_{ij}^{(\beta)}, \quad i \ge j.$$

Proof. As (12) then

$$\begin{array}{l}
\dfrac{\alpha_j}{\alpha_i} \le \dfrac{\beta_j}{\beta_i}, \quad i \le j, \\
\alpha_j \mu_i \beta_i \le \beta_j \mu_i \alpha_i, \quad i \le j, \\
\alpha_j(\mu_i\beta_i + \mu_j\beta_j) \le \beta_j(\mu_i\alpha_i + \mu_j\alpha_j), \quad i \le j, \\
\dfrac{\alpha_j}{\mu_i\alpha_i + \mu_j\alpha_j} \le \dfrac{\beta_j}{\mu_i\beta_i + \mu_j\beta_j}, \quad i \le j, \\
\sigma_{ij}^{(\alpha)} \le \sigma_{ij}^{(\beta)}, \quad i \le j.
\end{array}$$

We prove the second inequality of Lemma 6 in a similar way. □

Lemma 7. *If α, β satisfy (4), then*

$$\overline{T}^{DPS}(\alpha) \le \overline{T}^{DPS}(\beta),$$

when the elements of vector $y = \underline{1}'(E - B^{(\alpha)})^{-1}M$ are such that $y_1 \ge y_2 \ge \ldots \ge y_M$.

Proof. Let us denote $B^{(g)} = A^{(g)} + D^{(g)}$, $g = \alpha, \beta$. Then as (10)

$$\overline{T}^{DPS}(g) = \lambda^{-1}\underline{1}'(E - B^{(g)})^{-1}\left[\rho_1, \ldots, \rho_M\right]^T, \quad g = \alpha, \beta.$$

Following the method described in [10] we get the following.

$$\begin{aligned}
\overline{T}^{DPS}(\alpha) - \overline{T}^{DPS}(\beta) &= \lambda^{-1}\underline{1}'(E - B^{(\alpha)})^{-1}\left[\rho_1, \ldots, \rho_M\right]^T - \lambda^{-1}\underline{1}'(E - B^{(\beta)})^{-1}\left[\rho_1, \ldots, \rho_M\right]^T = \\
&= \lambda^{-1}\underline{1}'((E - B^{(\alpha)})^{-1} - (E - B^{(\beta)})^{-1})\left[\rho_1, \ldots, \rho_M\right]^T = \\
&= \lambda^{-1}\underline{1}'((E - B^{(\alpha)})^{-1}(B^{(\alpha)} - B^{(\beta)})(E - B^{(\beta)})^{-1})\left[\rho_1, \ldots, \rho_M\right]^T.
\end{aligned}$$

Let us denote M as a diagonal matrix $M = diag(\mu_1, \ldots, \mu_M)$ and

$$y = \underline{1}'(E - B^{(\alpha)})^{-1}M. \tag{13}$$

Then

$$\begin{aligned}
\overline{T}^{DPS}(\alpha) - \overline{T}^{DPS}(\beta) &= \underline{1}'(E - B^{(\alpha)})^{-1}MM^{-1}(B^{(\alpha)} - B^{(\beta)})\overline{T}^{(\beta)} = \\
&= yM^{-1}(B^{(\alpha)} - B^{(\beta)})\overline{T}^{(\beta)} = \\
&= \sum_{i,j}\left(\frac{y_j}{\mu_j}\sigma_{ji}^{(\alpha)} + \frac{y_i}{\mu_i}\sigma_{ij}^{(\alpha)} - \left(\frac{y_j}{\mu_j}\sigma_{ji}^{(\beta)} + \frac{y_i}{\mu_i}\sigma_{ij}^{(\beta)}\right)\right)\overline{T}_j^{(\beta)} = \\
&= \sum_{i,j}\left(y_j\left(\frac{\sigma_{ji}^{(\alpha)}}{\mu_j} - \frac{\sigma_{ji}^{(\beta)}}{\mu_j}\right) + \frac{y_i}{\mu_i}(\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)})\right)\overline{T}_j^{(\beta)}.
\end{aligned}$$

As (11):

$$\frac{\sigma_{ji}^{(g)}}{\mu_j} = \frac{1}{\mu_i\mu_j} - \frac{\sigma_{ij}^{(g)}}{\mu_i}, \quad g = \alpha, \beta,$$

then

$$\begin{aligned}
\overline{T}^{DPS}(\alpha) - \overline{T}^{DPS}(\beta) &= \sum_{i,j} \left(-y_j \left(\frac{\sigma_{ij}^{(\alpha)}}{\mu_i} - \frac{\sigma_{ij}^{(\beta)}}{\mu_i} \right) + \frac{y_i}{\mu_i}(\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \right) \overline{T}_j^{(\beta)} = \\
&= \sum_{i,j} \left(-\frac{y_j}{\mu_i} \left(\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)} \right) + \frac{y_i}{\mu_i}(\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)}) \right) \overline{T}_j^{(\beta)} = \\
&= \sum_{i,j} \left(\left(\sigma_{ij}^{(\alpha)} - \sigma_{ij}^{(\beta)} \right) (y_i - y_j) \frac{1}{\mu_i} \right) \overline{T}_j^{(\beta)}.
\end{aligned}$$

Using Lemma 6 we get $\left(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)}\right)(y_i - y_j)$ is negative for $i, j = 1, \dots, M$ when $y_1 \geq y_2 \geq \dots \geq y_M$. This proves the statement of Lemma 7. □

Lemma 8. *Vector y given by (13) satisfies*

$$y_1 \geq y_2 \geq \dots \geq y_M,$$

if the following is true:

$$\frac{\mu_{i+1}}{\mu_i} \leq 1 - \rho,$$

for every $i = 1, \dots, M$.

Proof. The proof could be found in the appendix. □

Remark 9. *For the job classes such as $\frac{\mu_{i+1}}{\mu_i} > 1-\rho$ we prove that to make $y_i \geq y_{i+1}$ it is sufficient to set the weights of these classes equal, $\alpha_{i+1} = \alpha_i$.*

Combining the results of Lemmas 5, 6, 7 and 8 we prove the statement of the Theorem 1. Remark 9 gives the Remark 2 after Theorem 1.

Proposition 10. *The result of Theorem 1 is extended to the case when $\lambda_i \neq 1$.*

Proof. Let us first consider the case when all $\lambda_i = q$, $i = 1, \dots, M$. It can be shown that for this case the proof of Theorem 1 is equivalent to the proof of the same Theorem but for the new system with $\lambda_i^* = 1$, $\mu_i^* = q\mu_i$, $i = 1, \dots, M$. For this new system the results of Theorem 1 is evidently true and restriction (5) is not changed. Then, Theorem 1 is true for the initial system as well.

If λ_i are rational, then they could be written in $\lambda_i = \frac{p_i}{q}$, where p_i and q are positive integers. Then each class can be presented as p_i classes with equal means $1/\mu_i$ and intensity $1/q$. So, the DPS system can be considered as the DPS system with $p_1 + \dots + p_K$ classes with the same arrival rates $1/q$. The result of Theorem 1 is extended on this case.

If λ_i, $i = 1, \dots, M$ are positive and real we apply the previous case of rational λ_i and use continuity. □

4 Numerical results

Let us consider a DPS system with 3 classes. Let us consider the set of normalized weights vectors $g(x) = (g_1(x), g_2(x), g_3(x))$, $\sum_{i=1}^3 g_i(x) = 1$, $g_i(x) = x^{-i}/(\sum_{i=1}^3 x^{-i})$, $x > 1$. Every point $x > 1$ denotes a weight vector. Vectors $g(x), g(y)$ satisfy property (4) when $1 < y \leq x$, namely $g_{i+1}(x)/g_i(x) \leq g_{i+1}(y)/g_i(y)$, $i = 1, 2$, $1 < y \leq x$. On Figures 1, 2 we plot $\overline{T}^{DPS}$ with weights vectors $g(x)$ as a function of x, the expected sojourn times $\overline{T}^{PS}$ for the PS policy and $\overline{T}^{opt}$ for the optimal $c\mu$-rule policy.

On Figure 1 we plot the expected sojourn time for the case when condition (5) is satisfied for three classes. The parameters are: $\lambda_i = 1$, $i = 1, 2, 3$, $\mu_1 = 160$, $\mu_2 = 14$, $\mu_3 = 1.2$, then $\rho = 0.911$.

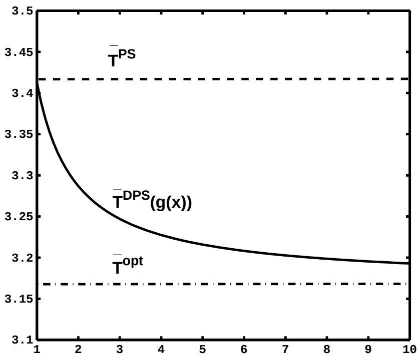

Figure 1: $\overline{T}^{DPS}(g(x))$, $\overline{T}^{PS}$, $\overline{T}^{opt}$ functions, condition satisfied.

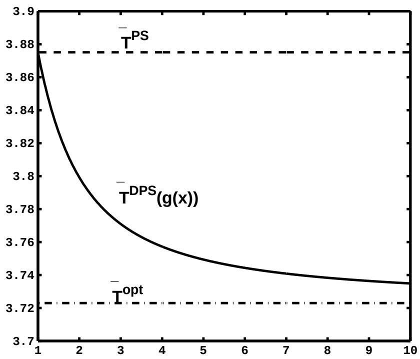

Figure 2: $\overline{T}^{DPS}(g(x))$, $\overline{T}^{PS}$, $\overline{T}^{opt}$ functions, condition not satisfied

On Figure 2 we plot the expected sojourn time for the case when condition (5) is *not* satisfied for three classes. The parameters are: $\lambda_i = 1$, $i = 1,2,3$, $\mu_1 = 3.5$, $\mu_2 = 3.2$, $\mu_3 = 3.1$, then $\rho = 0.92$. One can see that $\overline{T}^{DPS}(g(x)) \leq \overline{T}^{DPS}(g(y))$, $1 < y \leq x$ even when the restriction (5) is not satisfied.

5 Conclusion

We study the DPS policy with exponential job size distributions. One of the main problems studying DPS is the expected sojourn time minimization according to the weighs selection. In the present paper we compare two DPS policies with different weights. We show that the expected sojourn time is smaller for the policy with the weigh vector closer to the optimal policy vector, provided by $c\mu$-rule. So, we prove the monotonicity of the expected sojourn time for the DPS policy according to the weight vector selection.

The result is proved with some restrictions on system parameters. The found restrictions on the system parameters are such that the result is true for systems such as the mean values of the job class size distributions are very different from each other. We found, that to prove the main result it is sufficient to give the same weights to the classes with similar means. The found restriction is a sufficient and not a necessary condition on a system parameters. When the load of the system decreases, the condition becomes less strict.

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6 Appendix

In the following proof in the notations we do not use the dependency of the parameters on g to simplify the notations. We consider that vector $g \in G$, or $g_1 \geq g_2 \ldots \geq g_M$. To simplify the notations let us use $\sum_k$ instead of $\sum_{k=1}^M$.

Lemma 8. Vector $y = \underline{1}'(E-B)^{-1}M$ satisfies

$$y_1 \geq y_2 \geq \ldots \geq y_M,$$

if the following is true:

$$\frac{\mu_{i+1}}{\mu_i} \le 1-\rho,$$

for every $i = 1, \ldots, M$.

Proof. Using the results of the following Lemmas we prove the statement of Lemma 8 and give the proof for Remark 9. $\square$

Let us give the following notations

$$\tilde{\mu} = \mu^T (E-D)^{-1}, \tag{14}$$
$$\tilde{A} = M^{-1} A M (E-D)^{-1}. \tag{15}$$

Let us notice the following

$$\begin{aligned} (E-D)_j^{-1} &= \frac{1}{1-\sum_k \frac{g_k}{\mu_j g_j + \mu_k g_k}} = \frac{1}{1-\rho+\sum_k \frac{\mu_j g_j}{\mu_k(\mu_j g_j+\mu_k g_k)}} > 0, \quad j = 1, \ldots, M, \\ \tilde{A}_{ij} &= \frac{\frac{\mu_j g_j}{\mu_i(\mu_i g_i + \mu_j g_j)}}{1-\sum_k \frac{g_k}{\mu_j g_j + \mu_k g_k}} = \frac{\frac{\mu_j g_j}{\mu_i(\mu_i g_i + \mu_j g_j)}}{1-\rho+\sum_k \frac{\mu_j g_j}{\mu_k(\mu_j g_j+\mu_k g_k)}} > 0, \quad i,j = 1, \ldots, M \end{aligned}$$

Let us give the following notation

$$f(x) = \sum_k \frac{x}{\mu_k(x+\mu_k g_k)}.$$

Then

$$\begin{aligned} (E-D)_j^{-1} &= \frac{1}{1-\rho+f(\mu_j g_j)}, \quad j = 1, \ldots, M, \\ \tilde{A}_{ij} &= \frac{\mu_j g_j}{\mu_i(\mu_i g_i + \mu_j g_j)(1-\rho+f(\mu_j g_j))}, \quad i,j = 1, \ldots, M. \end{aligned}$$

Let us first prove additional Lemma.

Lemma 11. *Matrix*

$$\tilde{A} = M^{-1} A M (E-D)^{-1}$$

is a positive contraction.

Proof. Matrix $\tilde{A}$ is a positive operator as elements of matrices M and A are positive and elements of matrix $(E-D)^{-1}$ are positive. Let $\Omega = \{X | x_i \ge 0, \ i = 1, \ldots, M\}$. If $X \in \Omega$, then $\tilde{A}X \in \Omega$. Then to prove that matrix $\tilde{A}$ is a contraction it is enough to show that

$$\exists\, q, \quad 0 < q < 1, \quad ||\tilde{A}X|| \le q||X||, \quad \forall\ X \in \Omega. \tag{16}$$

As $X \in \Omega$, then we can take $||X|| = \underline{1}'X = \sum_i x_i$. Then

$$\begin{aligned} \underline{1}'\tilde{A}X &= \sum_j x_j \sum_i \tilde{A}_{ij} = \sum_j x_j \frac{\sum_i \frac{\mu_j g_j}{\mu_i(\mu_j g_j + \mu_i g_i)}}{(1-\rho+g(\mu_j g_j))} = \\ &= \sum_j x_j \frac{f(\mu_j g_j)}{1-\rho+f(\mu_j g_j)} = \sum_j x_j \left(1 - \frac{1-\rho}{1-\rho+f(\mu_j g_j)}\right) = \\ &= \sum_j x_j - (1-\rho)\sum_j \frac{x_j}{1-\rho+f(\mu_j g_j)}. \end{aligned}$$

Let us find the value of q, which satisfies condition (16).

$$\begin{aligned}
&\underline{1}'\tilde{A}X \le \underline{1}'X,\\
&\sum_j x_j - (1-\rho)\sum_j \frac{x_j}{1-\rho+f(\mu_j g_j)} \le q\sum_j x_j\\
&1-(1-\rho)\frac{\sum_j \frac{x_j}{1-\rho+f(\mu_j g_j)}}{\sum_j x_j} \le q.
\end{aligned}$$

As $f(\mu_j g_j) > 0$ then

$$0 < 1-(1-\rho)\frac{\sum_j \frac{x_j}{1-\rho+f(\mu_j g_j)}}{\sum_j x_j} < 1.$$

Let us define δ in the following way:

$$\delta = \frac{1}{1-\rho+\max_j f(\mu_j g_j)} < \frac{\sum_j \frac{x_j}{1-\rho+f(\mu_j g_j)}}{\sum_j x_j}.$$

Then

$$1-(1-\rho)\frac{\sum_j \frac{x_j}{1-\rho+f(\mu_j g_j)}}{\sum_j x_j} < 1-(1-\rho)\delta.$$

Let us notice that $\max_j f(\mu_j g_j)$ always exists as the values of $\mu_j g_j$, $j = 1,\dots,M$ are finite. Then we can select

$$q = 1-(1-\rho)\delta, \quad 0 < q < 1.$$

Which completes the proof. □

Lemma 12. *If*

$$y_1^{(0)} = [0,\dots,0], \tag{17}$$
$$y^{(n)} = \tilde{\mu} + y^{(n-1)}\tilde{A}, \quad n = 1,2,\dots, \tag{18}$$

then $y^{(n)} \to y$, when $n \to \infty$.

Proof. Let us present y in the following way. As $B = E - A - D$, then

$$\begin{aligned}
&y = \underline{1}(E-B)^{-1}M,\\
&yM^{-1}(E-D-A) = \underline{1},\\
&yM^{-1}(E-D) = -yM^{-1}A + \underline{1},\\
&y(E-D)^{-1}M = -yM^{-1}A(E-D)^{-1}M + \underline{1}(E-D)^{-1}M.
\end{aligned}$$

As matrixes D and M are diagonal, the $MD = DM$ and then

$$y = \mu^T(E-D)^{-1} + yM^{-1}AM(E-D)^{-1},$$

where $\mu = [\mu_1,\dots,\mu_M]$. According to notations (14) and (15) we have the following

$$y = \tilde{\mu} + y\tilde{A}.$$

Let us denote $y^{(n)} = [y_1^{(n)},\dots,y_1^{(n)}]$, $n = 0,1,2,\dots$ and let define $y_1^{(0)}$ and $y^{(n)}$ by (17) and (18). According to Lemma 11 reflection $\tilde{A}$ is a positive reflection and is a contraction. Also $\tilde{\mu}_i$ are positive. Then $y^{(n)} \to y$, when $n \to \infty$ and we prove the statement of Lemma 12. □

Lemma 13. *Let $y^{(n)}$ is defined by (18) and $y^{(0)}$ is given by (17), then*

$$y_1^{(n)} \geq y_2^{(n)} \geq \ldots \geq y_M^{(n)}, \quad n = 1, 2, \ldots \tag{19}$$

if $\frac{\mu_{i+1}}{\mu_i} \leq 1 - \rho$ for every $i = 1, \ldots, M$.

Proof. We prove the statement (19) by induction. For $y^{(0)}$ the statement (19) is true. Let us assume that (19) is true for the $(n-1)$ step, $y_1^{(n-1)} \geq y_2^{(n-1)} \geq \ldots \geq y_M^{(n-1)}$. To prove the induction statement we have to show that $y_1^{(n)} \geq y_2^{(n)} \geq \ldots \geq y_M^{(n)}$, which is equal to that $y_j^{(n)} \geq y_p^{(n)}$, if $j \leq p$. As

$$y_j^{(n)} = \tilde{\mu}_j + \sum_{i=1}^{M} y_i^{(n-1)} \tilde{A_{ij}},$$

then

$$\begin{aligned} y_j^{(n)} - y_p^{(n)} &= \tilde{\mu}_j + \sum_{i=1}^{M} y_i^{(n-1)} \tilde{A_{ij}} - \left(\tilde{\mu}_p + \sum_{i=1}^{M} y_i^{(n-1)} \tilde{A_{ip}} \right) = \\ &= \tilde{\mu}_j - \tilde{\mu}_p + \sum_{i=1}^{M} y_i^{(n-1)} (\tilde{A_{ij}} - \tilde{A_{ip}}). \end{aligned}$$

To show that $y_j^{(n)} - y_p^{(n)}$ we need to show that $\tilde{\mu}_j - \tilde{\mu}_p \geq 0$ and $\sum_{i=1}^{M} y_i^{(n-1)} (\tilde{A_{ij}} - \tilde{A_{ip}}) \geq 0$, when $j \leq p$. Let us show that to prove that $\sum_{i=1}^{M} y_i^{(n-1)} (\tilde{A_{ij}} - \tilde{A_{ip}}) \geq 0$, $j \leq p$ it is enough to prove that $\sum_{i=1}^{r} (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, $j \leq p$, $r = 1, \ldots, M$. If we regroup this sum we can get the following

$$\begin{aligned} \sum_{i=1}^{M} y_i^{(n-1)} (\tilde{A_{ij}} - \tilde{A_{ip}}) &= \\ \sum_{i=1}^{M} (y_i^{(n-1)} - y_{i+1}^{(n-1)} + y_{i+1}^{(n-1)} - \ldots - y_M^{(n-1)} + y_M^{(n-1)}) (\tilde{A_{ij}} - \tilde{A_{ip}}) &= \\ = \sum_{i=1}^{M-1} (y_i^{(n-1)} - y_{i+1}^{(n-1)}) \left[(\tilde{A}_{1j} - \tilde{A}_{1p}) + (\tilde{A}_{2j} - \tilde{A}_{2p}) + \ldots + (\tilde{A}_{ij} - \tilde{A}_{ip}) \right] &+ \\ + y_M^{(n-1)} ((\tilde{A}_{1j} - \tilde{A}_{1p}) + \ldots + (\tilde{A}_{(M-1)j} - \tilde{A}_{(M-1)p}) + (\tilde{A}_{Mj} - \tilde{A}_{Mp})) &= \\ = \sum_{i=1}^{M-1} (y_i^{(n-1)} - y_{i+1}^{(n-1)}) \sum_{k=1}^{r} (\tilde{A}_{kj} - \tilde{A}_{kp}) + y_M^{(n-1)} \sum_{k=1}^{M} (\tilde{A}_{kj} - \tilde{A}_{kp}). \end{aligned}$$

As $y_i^{(n-1)} \geq y_{i+1}^{(n-1)}$, $i = 1, \ldots, M$, according to the induction step, then to show that $\sum_{i=1}^{M} y_i^{(n-1)} (\tilde{A_{ij}} - \tilde{A_{ip}}) \geq 0$, $j \leq p$ it is enough to show that $\sum_{i=1}^{r} (\tilde{A}_{ij} - \tilde{A}_{ip}) \geq 0$, $j \leq p$, $r = 1, \ldots, M$. We show this in Lemma 15. In Lemma 14 we show $\tilde{\mu}_j \geq \tilde{\mu}_p$, $j \leq p$, when $\frac{\mu_{i+1}}{\mu_i} \leq 1 - \rho$ for every $i = 1, \ldots, M$. Then we prove the induction statement and so prove the statement of Lemma 13. □

Lemma 14.

$$\tilde{\mu}_1 \geq \tilde{\mu}_2 \ldots \geq \tilde{\mu}_M, \tag{20}$$

if $\frac{\mu_{i+1}}{\mu_i} \leq 1 - \rho$ for every $i = 1, \ldots, M$.

Proof. Let us compare $\tilde{\mu}_j$ and $\tilde{\mu}_p$, $j \leq p$. If $\mu_j = \mu_p$ and $g_j = g_p$, then $\tilde{\mu}_j = \tilde{\mu}_p$ and (20) is satisfied. Let us denote

$$f_2(x) = \sum_k \frac{g_k}{x + \mu_k g_k},$$

which has the following properties

$$0 < f_2(x) < \rho. \tag{21}$$

Then

$$\tilde{\mu}_i = \frac{\mu_i}{1 - \sum_j \frac{g_j}{\mu_i g_i + \mu_j g_j}} = \frac{\mu_i}{1 - f_2(\mu_i g_i)}.$$

Let us find

$$\tilde{\mu}_j - \tilde{\mu}_p = \frac{\mu_j}{1 - f_2(\mu_j g_j)} - \frac{\mu_p}{1 - f_2(\mu_p g_p)} = \frac{\mu_j - \mu_p - (\mu_j f_2(\mu_p g_p) - \mu_p f_2(\mu_j g_j))}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_p g_p))}.$$

As (21) then

$$\mu_j f_2(\mu_p g_p) - \mu_p f_2(\mu_j g_j) < \mu_j \rho.$$

Then

$$\begin{aligned} \tilde{\mu}_j - \tilde{\mu}_p &> \frac{(\mu_j - \mu_p)}{(1 - f_2(\mu_j g_j)(1 - f_2(\mu_p g_p)))} \left(1 - \rho\left(\frac{\mu_j}{\mu_j - \mu_p}\right)\right) = \\ &= \frac{(\mu_j - \mu_p)}{(1 - f_2(\mu_j g_j)(1 - f_2(\mu_p g_p)))} \left(1 - \rho\left(\frac{1}{1 - \frac{\mu_p}{\mu_j}}\right)\right) \geq 0, \end{aligned}$$

when

$$\frac{\mu_p}{\mu_j} \leq 1 - \rho.$$

So, if $\frac{\mu_p}{\mu_j} \leq 1 - \rho$ and $g_j \geq g_p$, then $\tilde{\mu}_j \geq \tilde{\mu}_p$. Let us show that if $\mu_j > \mu_p$ and $g_j = g_p$, then $\tilde{\mu_j} \geq \tilde{\mu_p}$. In this case

$$\begin{aligned} \tilde{\mu}_j - \tilde{\mu}_p &= \frac{\mu_j}{1 - f_2(\mu_j g_j)} - \frac{\mu_p}{1 - f_2(\mu_p g_p)} = \\ &= \frac{\mu_j - \mu_p - (\mu_j f_2(\mu_p g_j) - \mu_p f_2(\mu_j g_j))}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_p g_j))} \\ &= \frac{\Delta_1}{(1 - f_2(\mu_j g_j))(1 - f_2(\mu_p g_j))}. \end{aligned}$$

Let us find when $\Delta_1 > 0$.

$$\begin{aligned} \Delta_1 &= \mu_j - \mu_p - \left(\mu_j \sum_{k=1}^{M} \frac{g_k}{\mu_p g_j + \mu_k g_k} - \mu_p \sum_{k=1}^{M} \frac{g_k}{\mu_j g_j + \mu_k g_k}\right) = \\ &= \mu_j - \mu_p - \left(\sum_{k=1}^{M} \frac{g_k(g_j(\mu_j^2 - \mu_p^2) + \mu_k g_k(\mu_j - \mu_p))}{(\mu_p g_j + \mu_k g_k)(\mu_p g_j + \mu_k g_k)}\right) \\ &= (\mu_j - \mu_p)\left(1 - \sum_{k=1}^{M} \frac{g_k(g_j(\mu_j + \mu_p) + \mu_k g_k)}{(\mu_p g_j + \mu_k g_k)(\mu_p g_j + \mu_k g_k)}\right). \end{aligned}$$

As

$$
\begin{aligned}
&0 < \mu_j\mu_p g_j^2, \quad k = 1,\dots,M,\\
&g_k\mu_k(g_j(\mu_j+\mu_p)+\mu_k g_k) < (\mu_j\mu_p g_j^2 + g_j(\mu_j+\mu_p)\mu_k g_k + \mu_k^2 g_k^2), \quad k=1,\dots,M\\
&g_k\mu_k(g_j(\mu_j+\mu_p)+\mu_k g_k) < (\mu_j g_j + \mu_k g_k)(\mu_p g_j + \mu_k g_k), \quad k=1,\dots,M\\
&\frac{g_k(g_j(\mu_j+\mu_p)+\mu_k g_k)}{(\mu_j g_j+\mu_k g_k)(\mu_p g_j+\mu_k g_k)} < \frac{1}{\mu_k}, \quad k=1,\dots,M.
\end{aligned}
$$

Then

$$\Delta_1 > (\mu_j - \mu_p)\left(1 - \sum_{k=1}^{M}\frac{1}{\mu_k}\right) = 1-\rho > 0.$$

Then we proved the following:

$$
\begin{aligned}
&\text{If} \quad \mu_j = \mu_p,\ g_j = g_p, \quad \text{then} \quad \tilde{\mu_j} = \tilde{\mu_p},\\
&\text{If} \quad \mu_j > \mu_p,\ g_j = g_p, \quad \text{then} \quad \tilde{\mu_j} > \tilde{\mu_p},\\
&\text{If} \quad \frac{\mu_p}{\mu_j} \le 1-\rho,\ g_j \ge g_p, \quad \text{then} \quad \tilde{\mu_j} \ge \tilde{\mu_p}.
\end{aligned}
$$

Setting $p = j+1$ and remembering that $\mu_1 \ge \ldots \ge \mu_M$, we get that $\tilde{\mu}_1 \ge \tilde{\mu}_2 \ldots \ge \tilde{\mu}_M$ is true when $\frac{\mu_{i+1}}{\mu_i} \le 1-\rho$ for every $i = 1,\dots,M$. That proves the statement of Lemma 14.

Returning back to the main Theorem 1, Lemma 14 gives condition (5) as a restriction on a system parameters.

Let us notice that for the job classes such for which the means are such as $\frac{\mu_{i+1}}{\mu_i} < 1-\rho$, if the weights given for these classes are equal, then still $\tilde{\mu}_i \ge \tilde{\mu}_{i+1}$. This condition gives us as a result Remark 9 and Remark 2. □

Lemma 15.

$$\sum_{i=1}^{r}\tilde{A}_{i1} \ge \sum_{i=1}^{r}\tilde{A}_{i2} \ge \ldots \ge \sum_{i=1}^{r}\tilde{A}_{iM}, \quad r = 1,\dots,M.$$

Proof. Let us remember $\tilde{A} = M^{-1}AM(E-D)^{-1}$. Then as $\rho = \sum_{k=1}^{M}\frac{1}{\mu_k}$, then

$$\sum_{i=1}^{r}\tilde{A}_{ij} = \frac{\sum_{i=1}^{r}\frac{\mu_j g_j}{\mu_i(\mu_j g_j+\mu_i g_i)}}{1-\sum_{k=1}^{M}\frac{g_k}{\mu_j g_j+\mu_k g_k}} = \frac{\sum_{i=1}^{r}\frac{\mu_j g_j}{\mu_i(\mu_j g_j+\mu_i g_i)}}{1-\rho+\sum_{k=1}^{M}\frac{\mu_j g_j}{\mu_k(\mu_j g_j+\mu_k g_k)}}$$

Let us define

$$f_3(x) = \frac{\sum_{i=1}^{r}\frac{x}{\mu_i(x+\mu_i g_i)}}{1-\rho+\sum_{k=1}^{M}\frac{x}{\mu_k(x+\mu_k g_k)}} = \frac{h_1(x)}{1-\rho+h_1(x)+h_2(x)},$$

where

$$
\begin{aligned}
h_1(x) &= \sum_{i=1}^{r}\frac{x}{\mu_i(x+\mu_i g_i)} > 0,\\
h_2(x) &= \sum_{j=r+1}^{M}\frac{x}{\mu_j(x+\mu_j g_j)} > 0.
\end{aligned}
$$

Let us show that $f_3(x)$ is increasing on x. For that it enough to show that $\frac{df_3(x)}{dx} \ge 0$. Let us consider

$$\frac{df_3(x)}{dx} = \frac{h_1'(x)(1-\rho)+h_1'(x)h_2(x)-h_1(x)h_2'(x)}{(1-\rho+h_1(x)+h_2(x))^2}$$

Since $h_1'(x) > 0$ and $1 - \rho > 0$:

$$\frac{df_3(x)}{dx} \geq 0 \quad \text{if} \quad h_1'(x)h_2(x) - h_1(x)h_2'(x) \geq 0.$$

Let us consider

$$\begin{aligned}
h_1'(x)h_2(x) - h_1(x)h_2'(x) &= \sum_{i=1}^{r} \frac{g_i}{(x+\mu_i g_i)^2} \sum_{k=r+1}^{M} \frac{x}{\mu_k(x+\mu_k g_k)} - \sum_{i=1}^{r} \frac{x}{\mu_i(x+\mu_i g_i)} \sum_{k=r+1}^{M} \frac{g_k}{(x+\mu_k g_k)^2} = \\
&= \sum_{i=1}^{r} \sum_{k=r+1}^{M} \left(\frac{g_i x}{(x+\mu_i g_i)^2 (x+\mu_k g_k)\mu_k} - \frac{g_k x}{\mu_i (x+\mu_i g_i)(x+\mu_k g_k)^2} \right) = \\
&= \sum_{i=1}^{r} \sum_{k=r+1}^{M} \frac{x}{(x+\mu_i g_i)(x+\mu_k g_k)} \left(\frac{g_i}{\mu_k(x+\mu_i g_i)} - \frac{g_k}{\mu_i(x+\mu_k g_k)} \right) = \\
&= \sum_{i=1}^{r} \sum_{k=r+1}^{M} \frac{x}{(x+\mu_i g_i)(x+\mu_k g_k)} \left(\frac{\mu_i g_i(x+g_k\mu_k) - \mu_k g_k(x+\mu_i g_i)}{\mu_i\mu_k(x+\mu_k g_k)(x+\mu_i g_i)} \right) = \\
&= \sum_{i=1}^{r} \sum_{k=r+1}^{M} \frac{x^2\,(\mu_i g_i - \mu_k g_k)}{(x+\mu_i g_i)^2 (x+\mu_k g_k)^2 \mu_k \mu_i} \geq 0,
\end{aligned}$$

Then $\frac{df_3(x)}{dx} \geq 0$ and $f_3(x)$ is an increasing function of x. As $\mu_j g_j \geq \mu_p g_p$, $j \leq p$, then we prove the statement of Lemma 15. $\square$

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